# Lazy Data Structures

# "Average" analysis of algorithms

(without sacrificing worst-case robustness)

① Amortized analysis ("average" in hindsight)

② Randomized algorithms ("average" over random bits)

Both are very practical, different perspectives

# Incrementing Binary Counters

i++;  //increment i

In bits: 0, 1, 10, 11, 100, 101, ...

Running time of i++?

# Incrementing Binary Counters

`i++;` // increment i

In bits: 0, 1, 10, 11, 100, 101, ...

---

11111111111111111111111111111111,

100000000000000000000000000000000

k bits ⇒ k flips (not constant time!)

---

from $i=1$ to $n$, how many total flips?

| | how often | total |
|---|---|---|
| • 1st bit (least significant) | every time | $n$ |
| • 2nd bit | every 2 times | $\lfloor n/2 \rfloor$ |
| • 3rd bit | every 4 times | $\lfloor n/4 \rfloor$ |
| $\vdots$ | | |
| • kth bit | every $2^{k-1}$ times | $\lfloor n/2^{k-1} \rfloor$ |

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$n$ increments $\Rightarrow$ $O(n)$ total time

each increment takes $O(1)$ <u>on average</u>

"$O(1)$ amortized time"

# Amortized Analysis

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"$T$ amortized time"

$\Rightarrow$ $n$ operations take $O(nT)$ time

Extends to multiple operations $OP_1, \sim, OP_k$

"$OP_1$ takes $T_1$ amortized time, $OP_2$ takes $\sim$,

$OP_k$ takes $T_k$ amortized time"

# Ways to do amortized analysis

- direct analysis (like above)
- credit schemes ("banker's method")
- potential function

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# Credit schemes

Idea:

Buy credits ahead of time to pay for work later

$$(\text{running time}) + O(1) \cdot (\underbrace{\Delta \text{ credit}}_{\text{(net increase)}}) \le \text{amortized time}$$

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---

## e.g. for binary counters, when incrementing:

0, 1, 10, 11, 100, 101, …

11111111111111111111111111111111

100000000000000000000000000000000

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put: 1 credit on first bit

$\frac{1}{2}$ credit on 2nd bit

$\frac{1}{4}$ credit on 3rd bit

\+ $\frac{1}{2^{i-1}}$ credits on ith bit

---

2 credits total

whenever we need to flip bit i, we have already saved 1 credit to pay for it.

# Credit schemes

## Idea:

Buy credits ahead of time to pay for work later

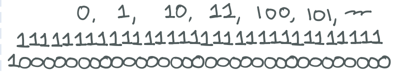
(running time) + $O(1) \cdot \underbrace{(\Delta \text{ credit})}_{\text{(net increase)}} \le$ amortized time

**Main rule:**

only spend credits that you already saved

e.g. for binary counters, when incrementing:

0, 1, 10, 11, 100, 101, ...



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put:

$$
\begin{array}{rl}
 & 1 \text{ credit on first bit} \\
 & 1/2 \text{ credit on 2nd bit} \\
 & 1/4 \text{ credit on 3rd bit} \\
 & \vdots \\
+ & 1/2^{i-1} \text{ credits on ith bit} \\
\hline
 & 2 \text{ credits total}
\end{array}
$$

whenever we need to flip bit $i$, we have already saved 1 credit to pay for it.

# Potential method

Invent "potential $\Phi$"


Amortized cost of operation:

$$(\text{real time}) + \underbrace{\Delta\Phi}_{\text{(change in potential)}}$$

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Amortized cost of operation:

$$(\text{real time}) + \Delta\Phi$$

(change in potential)

e.g. binary counter

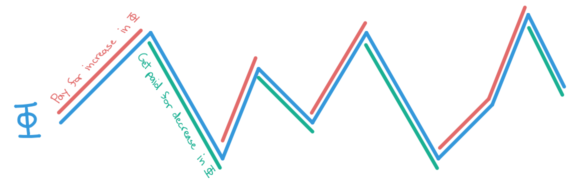
$$\Phi = \#\text{ bits set to } 1$$

then for each increment,

$$(\text{real time}) + \Delta\Phi \le O(1)$$

(each bit reset to 0 paid by $\Delta\Phi$)

"potential $\Phi$"




0, 1, 10, 11, 100, 101, ...

1111111111111111111111111111111

1000000000000000000000000000000000

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- credit schemes ("banker method")

"T amortized time"

$\Rightarrow$ n operations take $O(nT)$ time

Multiple operations $OP_1, \sim, OP_k$

"$OP_1$ takes $T_1$ amortized time, $OP_2$ take $\sim$ $OP_k$ takes $T_k$ amortized time"

put: 1 credit on first bit
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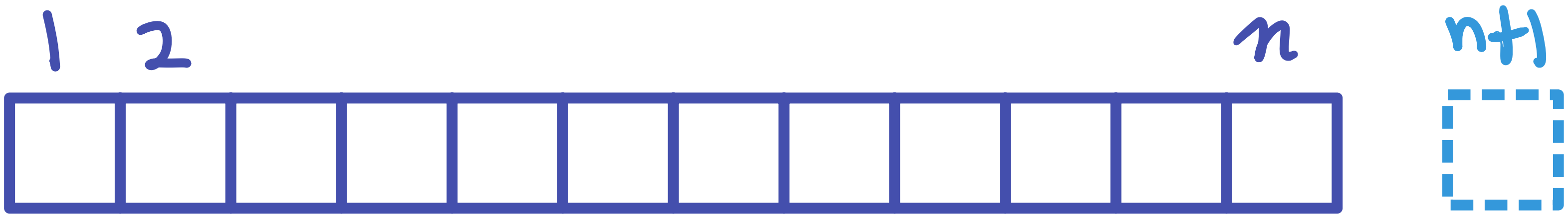
⋮
+ $\frac{1}{2^{i-1}}$ credits on ith bit

2 credits total

- potential function method

$\Phi$ = # bits set to 1

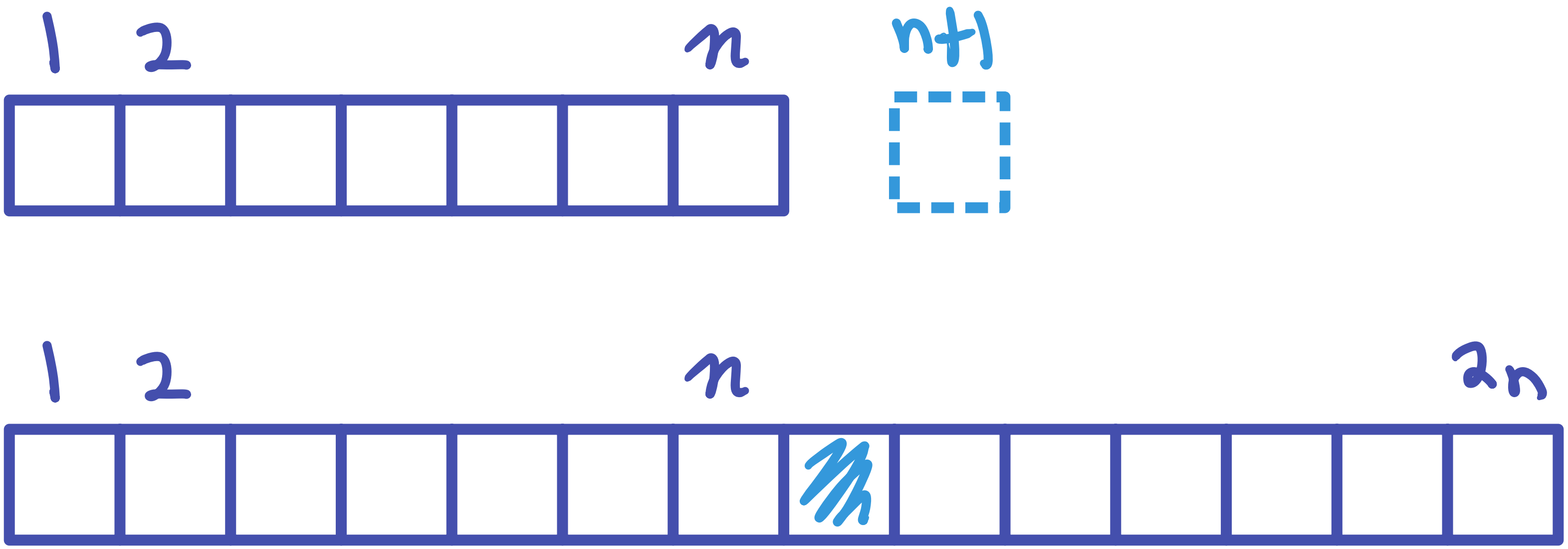
# Array-backed lists



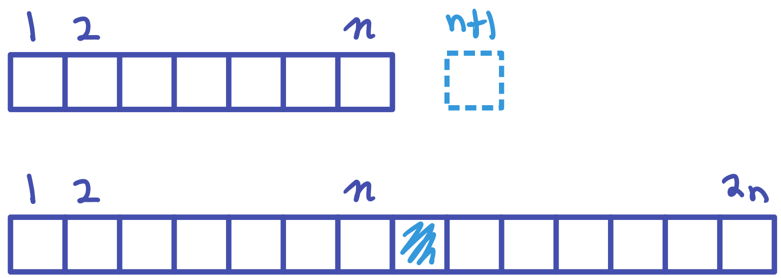

Good: O(1) access by index

Bad: O(n) append

# Doubling Trick



# Doubling Trick




**Theorem** w/ doubling trick, arraylist has

- lookup in $O(1)$
- append in $O(1)$ amortized

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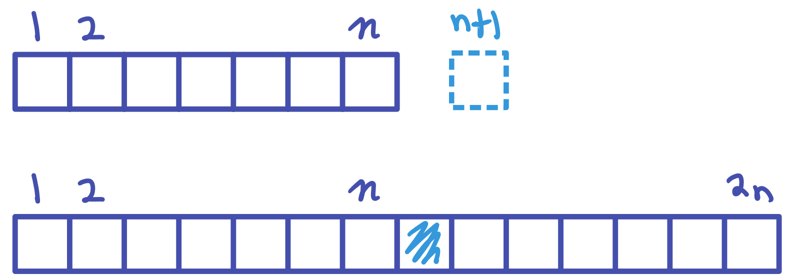
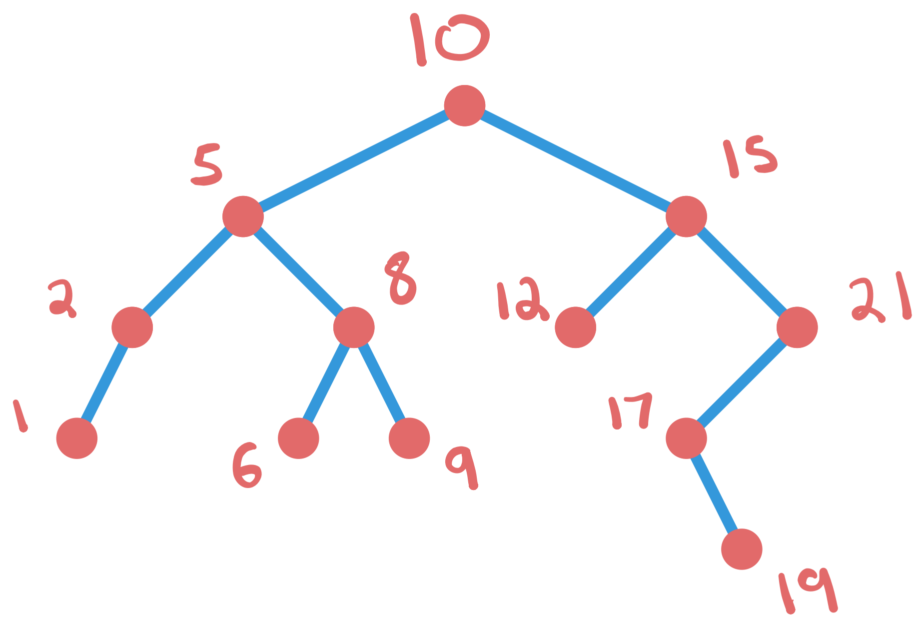
**Proof**

Doubling Trick



# Search trees

Organizes keys in a tree




**Invariant:**

a node's # is

- $\geq$ any # in left subtree
- $\leq$ any # in right subtree

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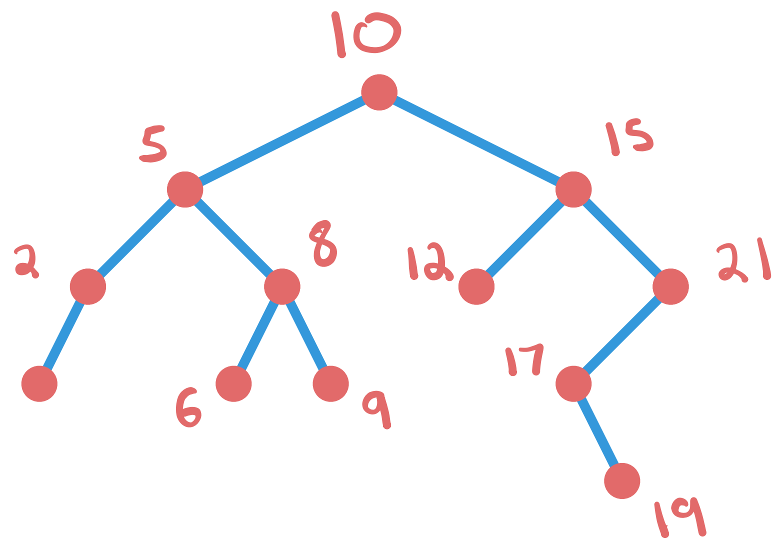
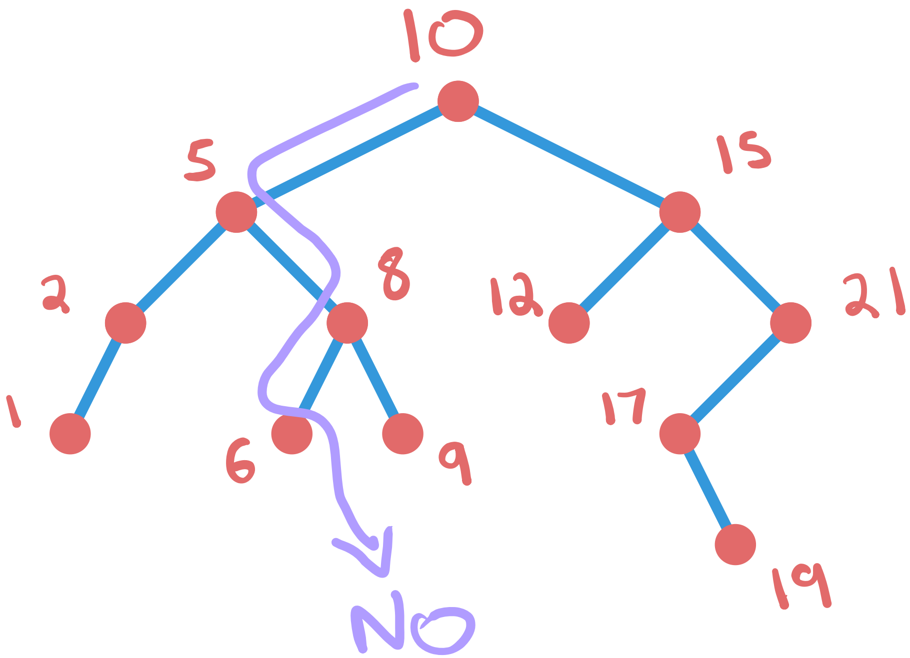


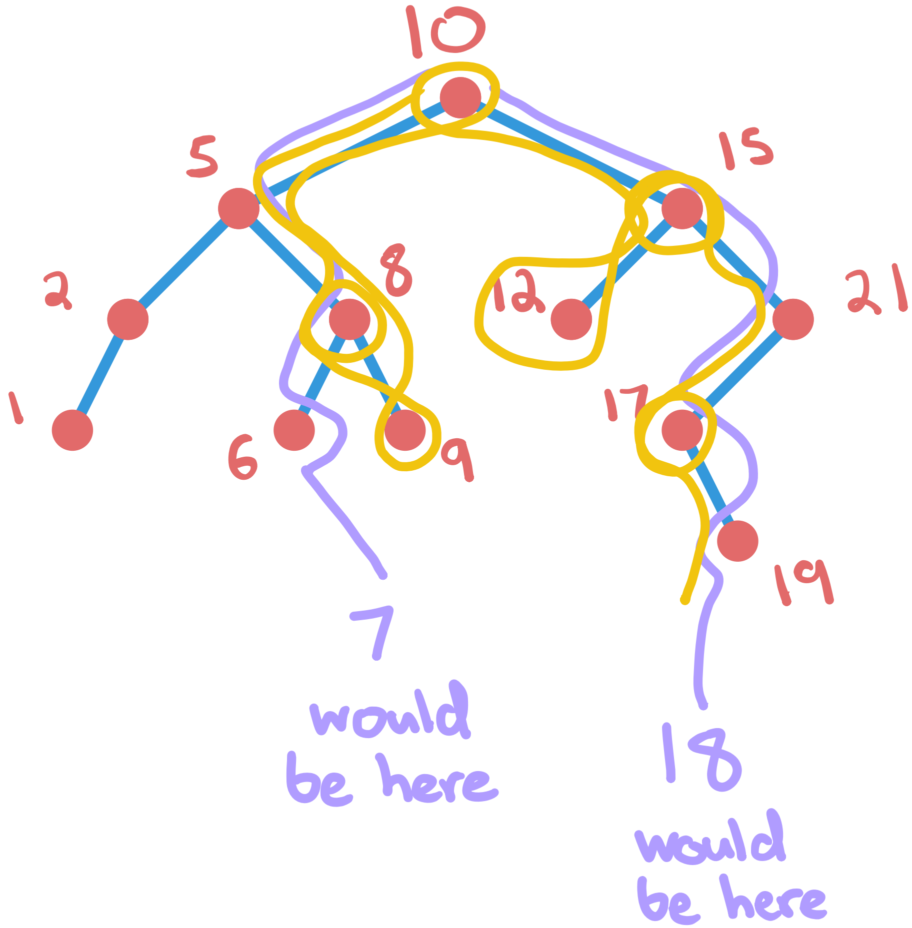

## Leads to operations like:

- is a given key in the set?
- what is the first key $\geq x$
- list all keys between $x$ and $y$
- how many keys are between $x$ and $y$?

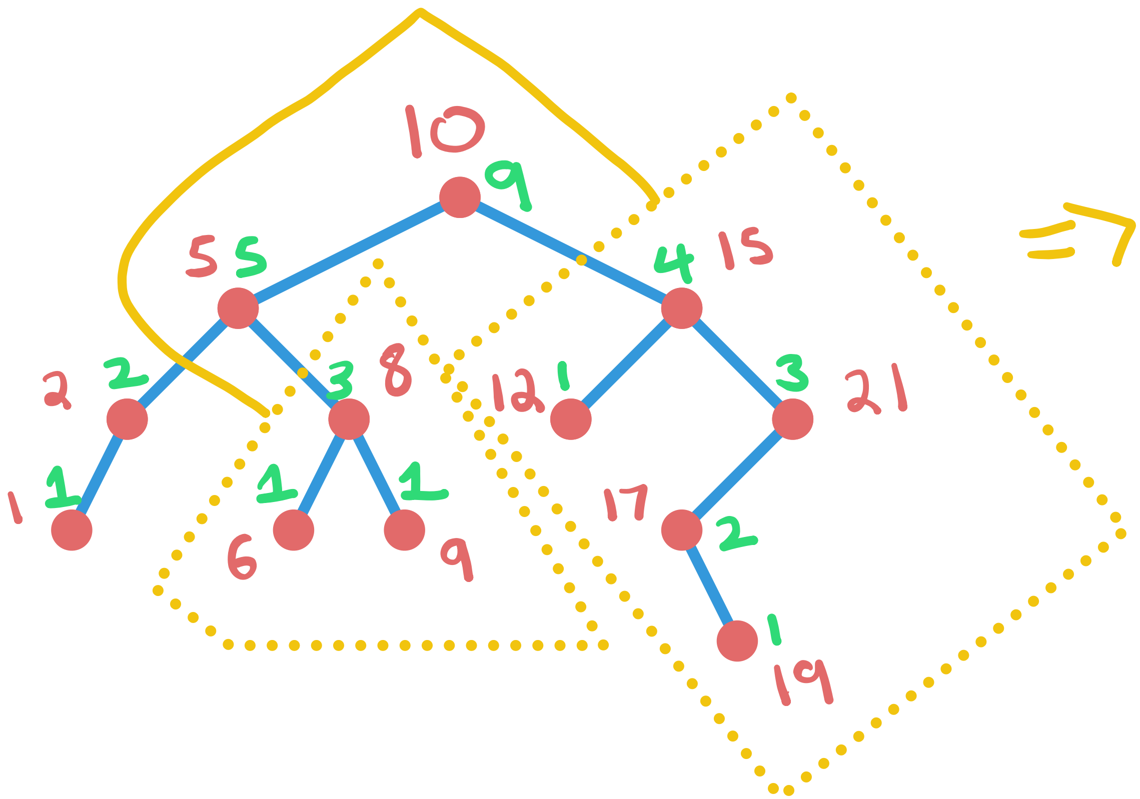
does the tree contain 7?



list all keys between 7 and 18

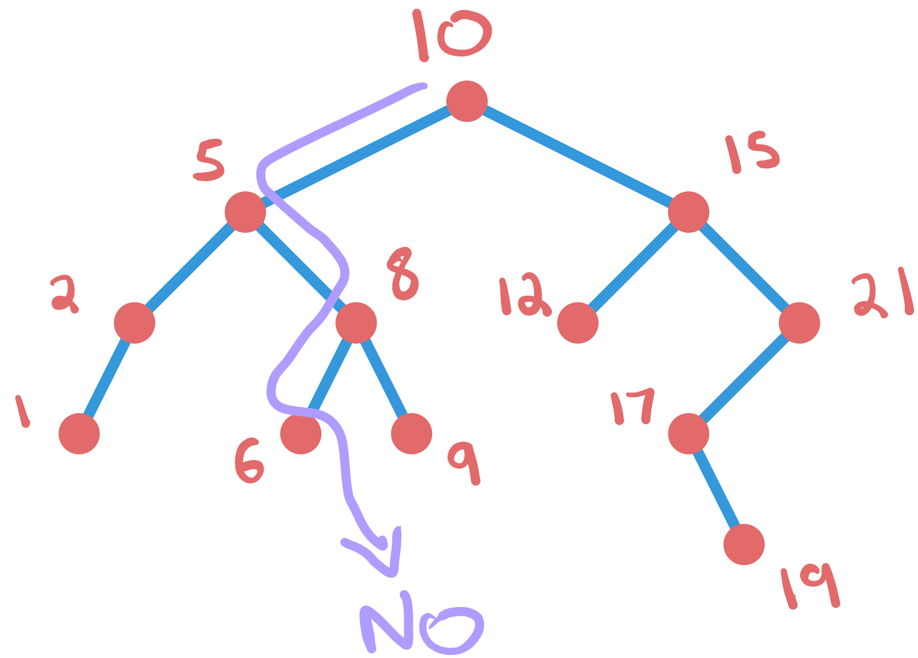


how many keys are between 5 and 25

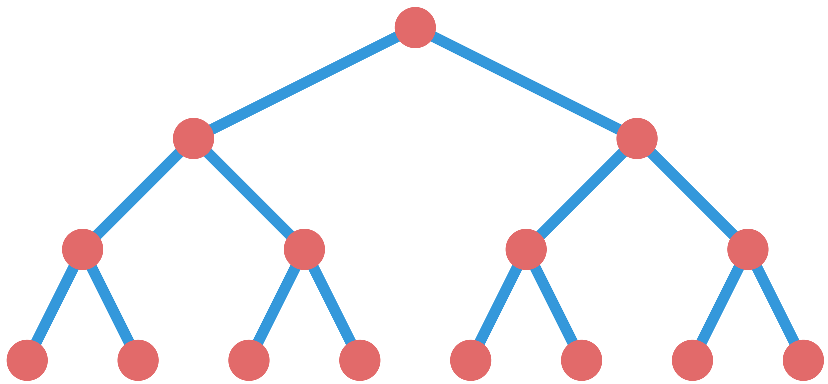


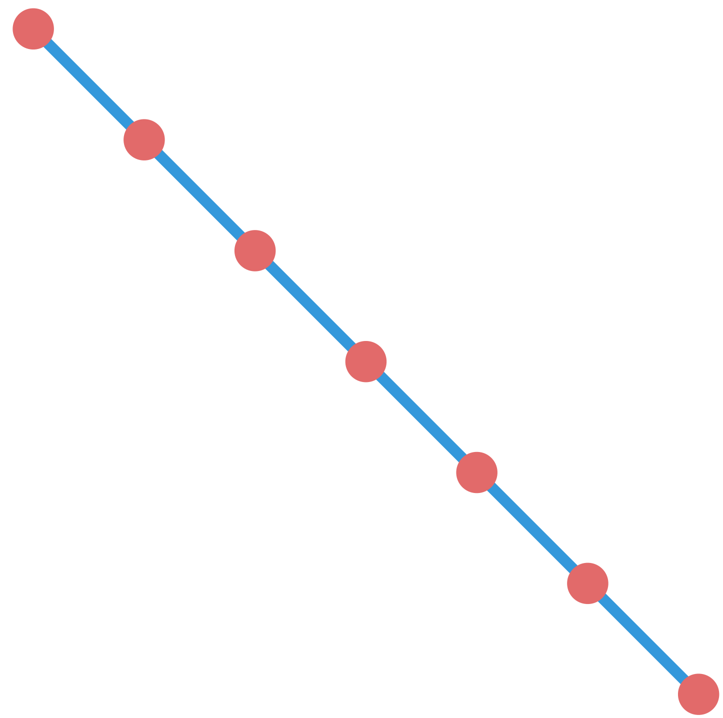

$\Rightarrow 9$

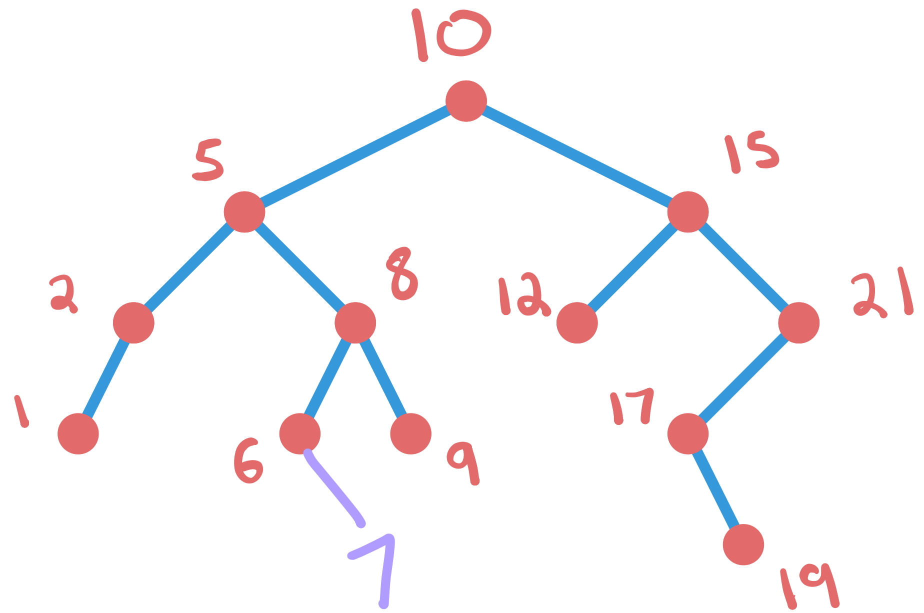
if we write down # nodes in each subtree, then faster




operations take time proportional to the height of a tree

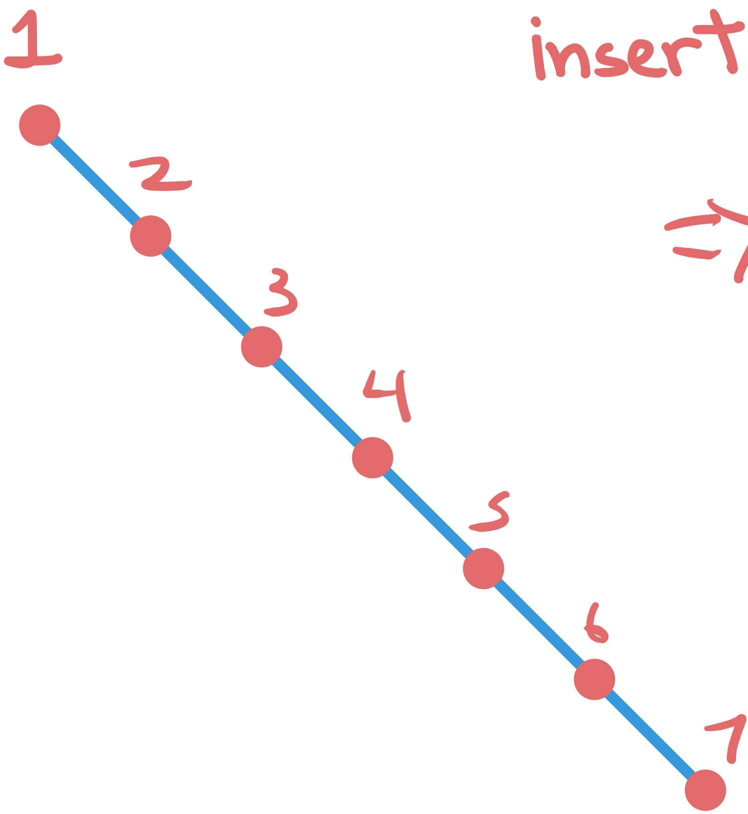






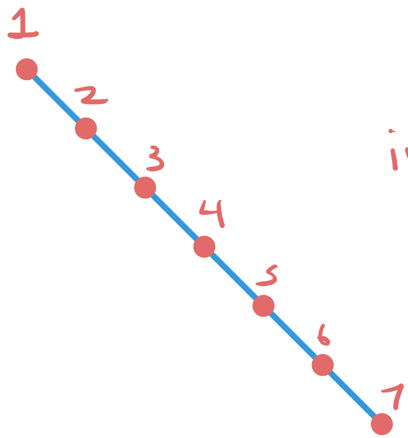

Most natural method for insertion is as leaves

(search for where the key would be, and put it there)




insert 1, 2, 3, 4, 5, 6, 7

$\Rightarrow$ imbalanced




insert 1, 2, 3, 4, 5, 6, 7

$\Rightarrow$ imbalanced

How to keep a tree balanced?

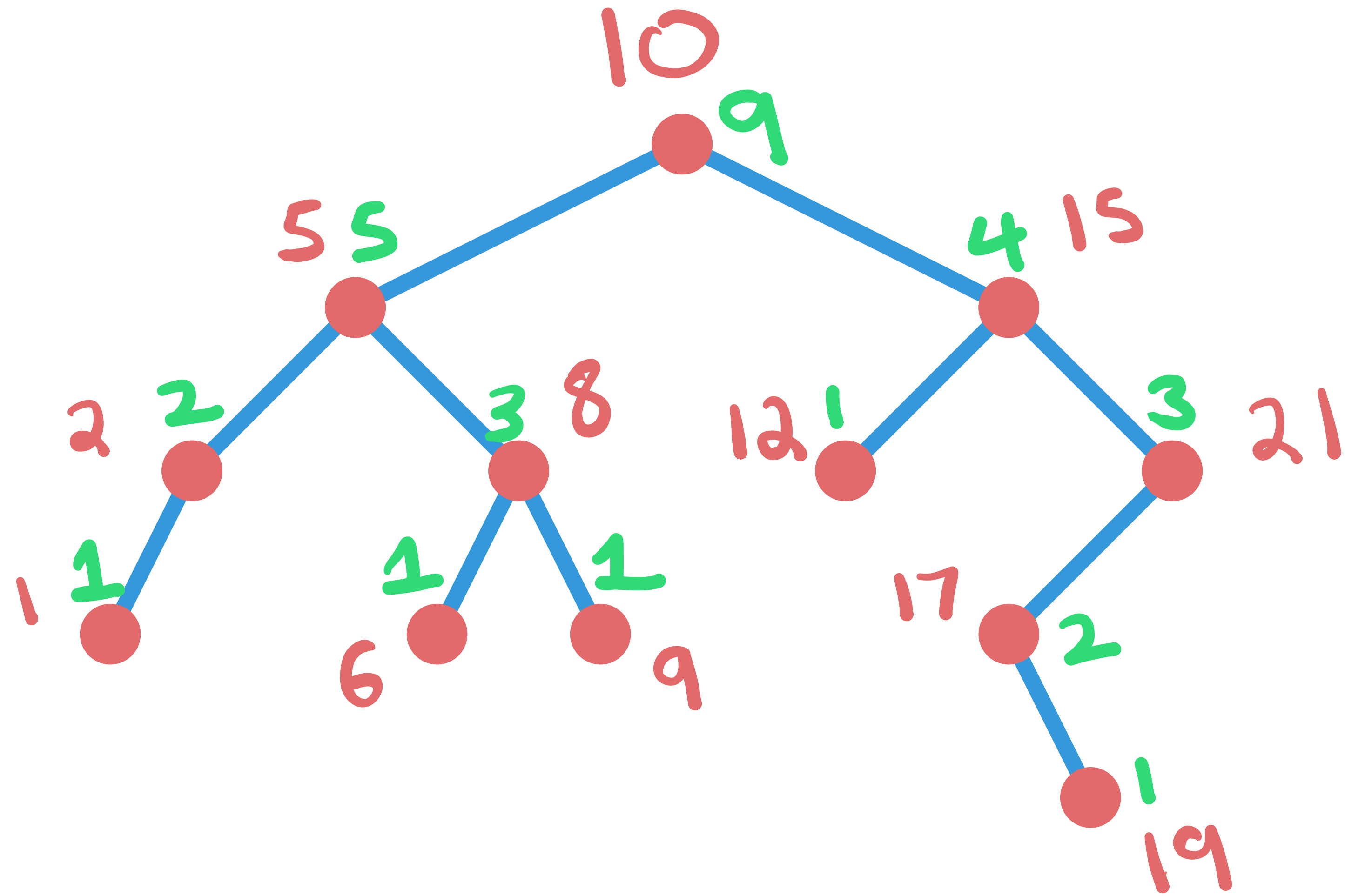
AVL trees

Red-black trees

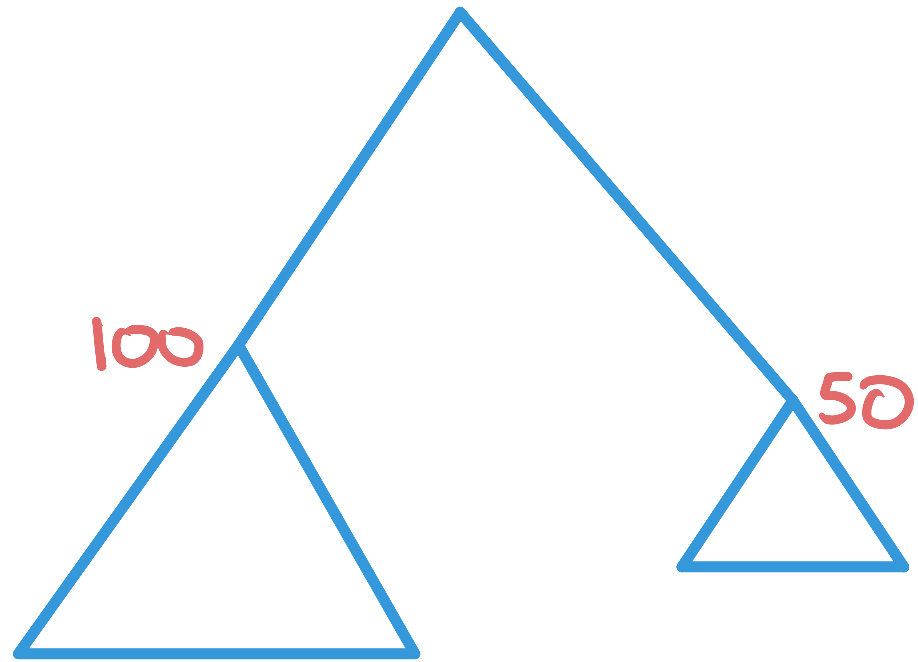
typically covered in UG data structures

Today we will be very lazy

write down # nodes in each subtree



If we detect big imbalance:



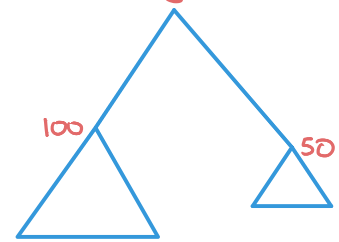

(# in one subtree) > 2 × (# in other subtree)

rebuild entire subtree optimally

(choose median to be root of subtree, recurse) (linear time)

**Lemma:** height is $O(\log n)$ (always)

If we detect big imbalance:




(# in one subtree) > 2 × (# in other subtree),

rebuild entire subtree tree optimally

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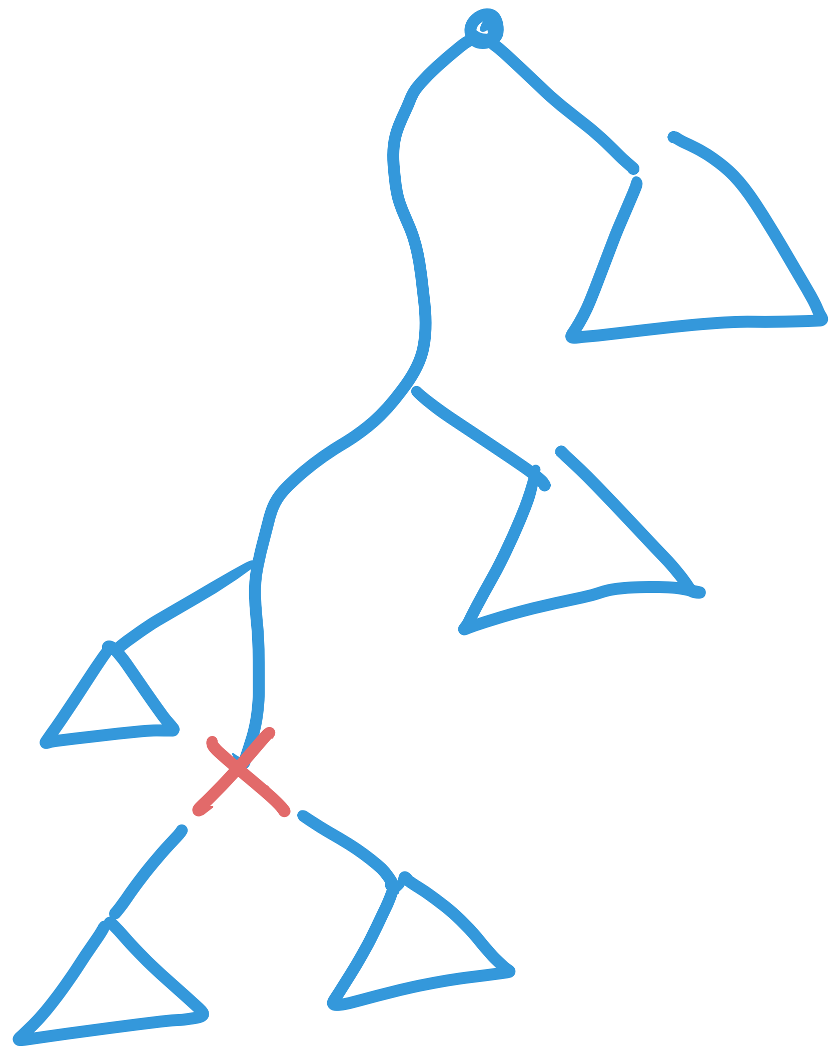
**Theorem** Insertion takes $O(\log n)$ amortized time

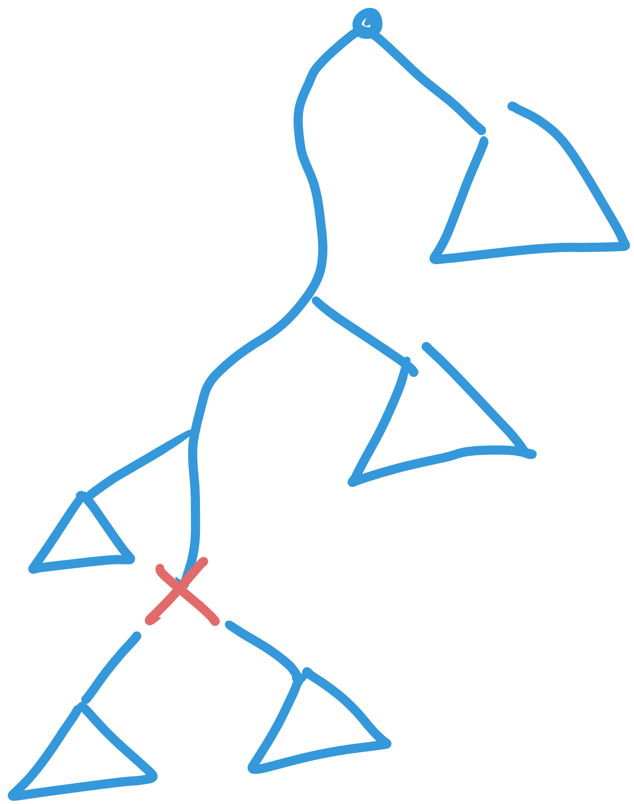
# Deletions (remove key from BST)

Old-fashioned way:

① search for key

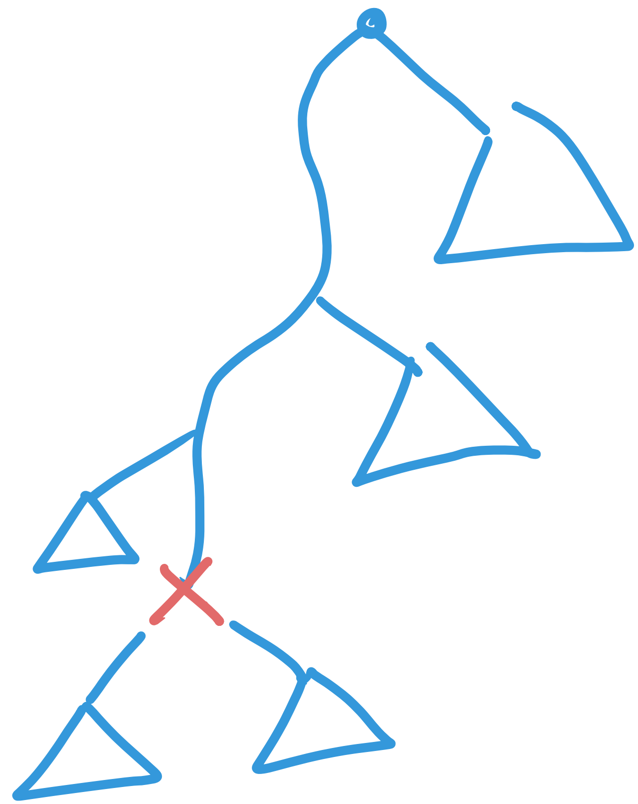
② If key is not a leaf:
chaos





Lazy idea: mark node as deleted

if $\geq 50\%$ marked for deletion, rebuild optimally



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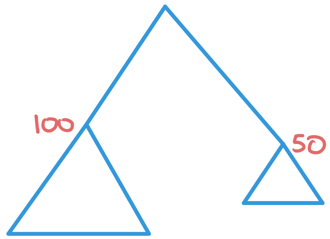


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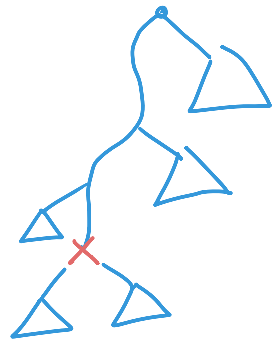

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**Theorem:** lazy insertion and lazy deletion take

$O(\log n)$ amortized time

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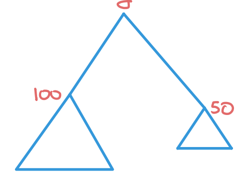
<u>Proof</u>

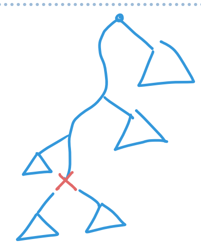
If we detect big imbalance:




$\left(\begin{smallmatrix}\#\text{ in one}\\\text{subtree}\end{smallmatrix}\right) > 2 \times \left(\begin{smallmatrix}\#\text{ in other}\\\text{subtree}\end{smallmatrix}\right)$,

rebuild entire subtree tree optimally

(choose median to be root of subtree, recurse) (linear time)



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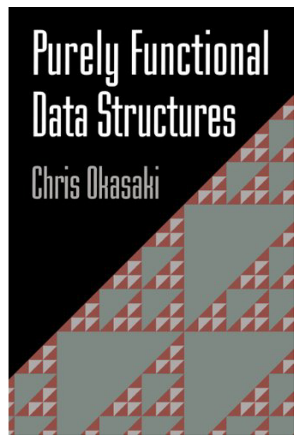
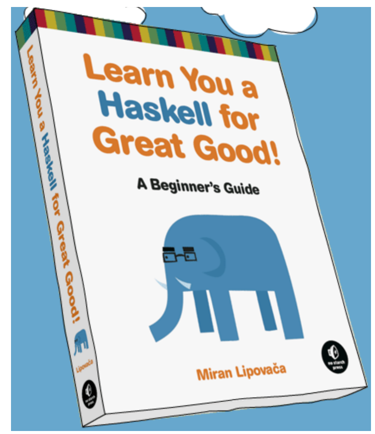
# Immutable data structures

Once something is written, never changes

Restrictions bring benefits:

- no side effects
- simplifies concurrency
- helps compiler



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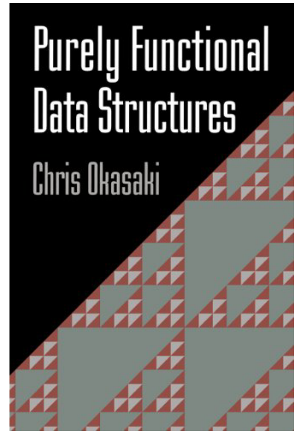
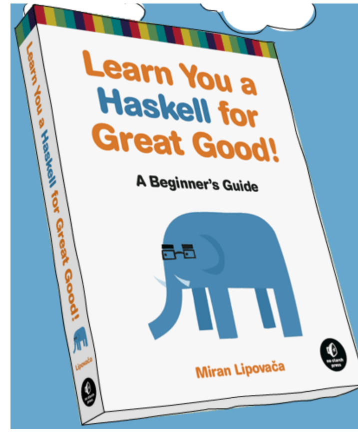
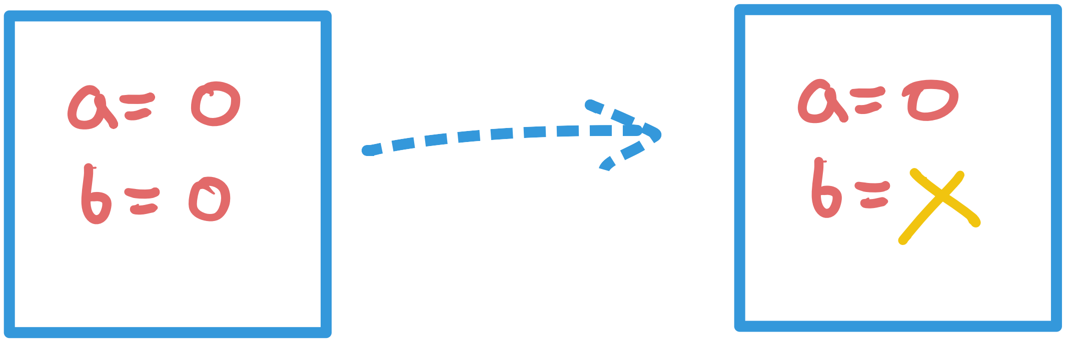


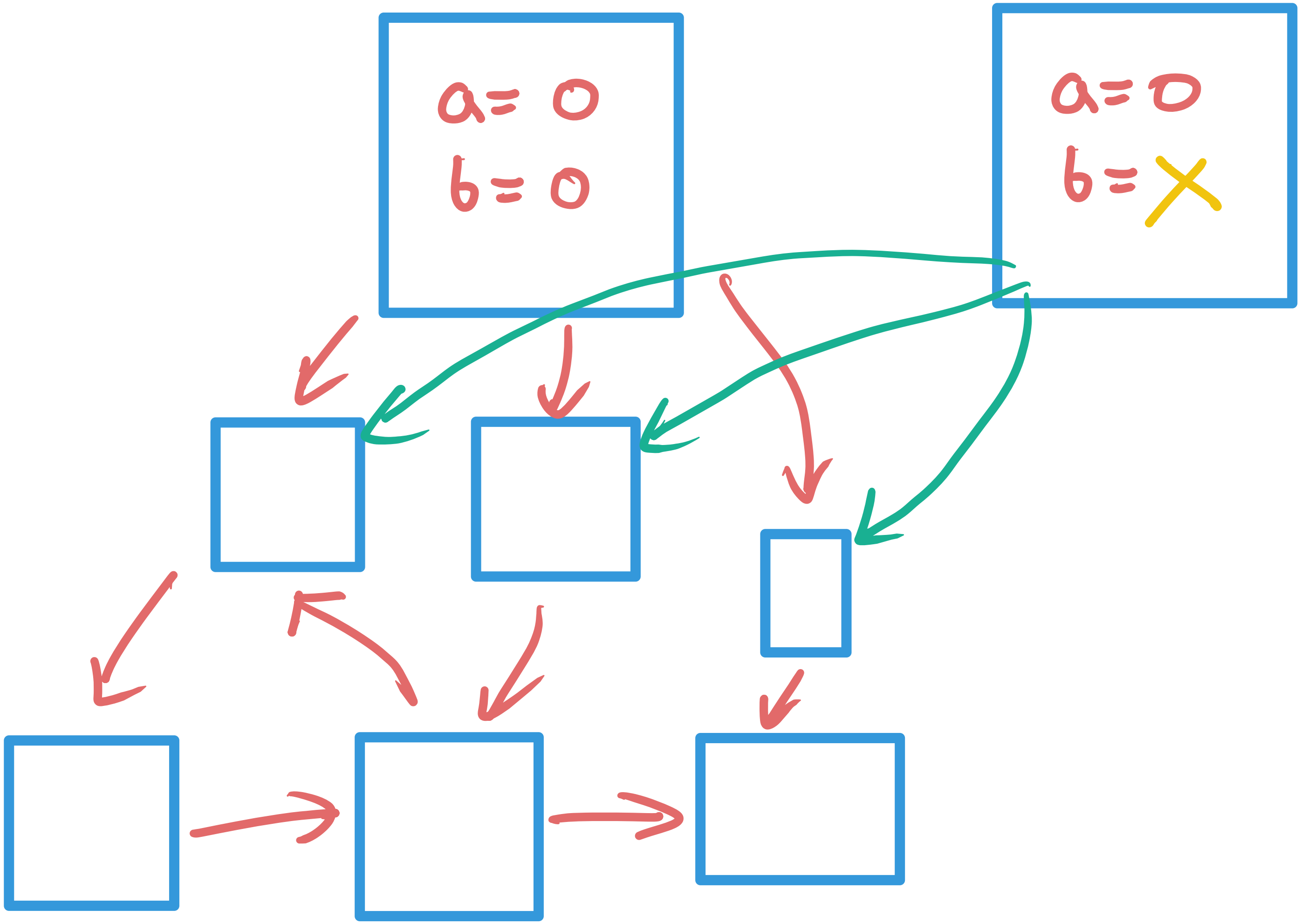

e.g. linked list

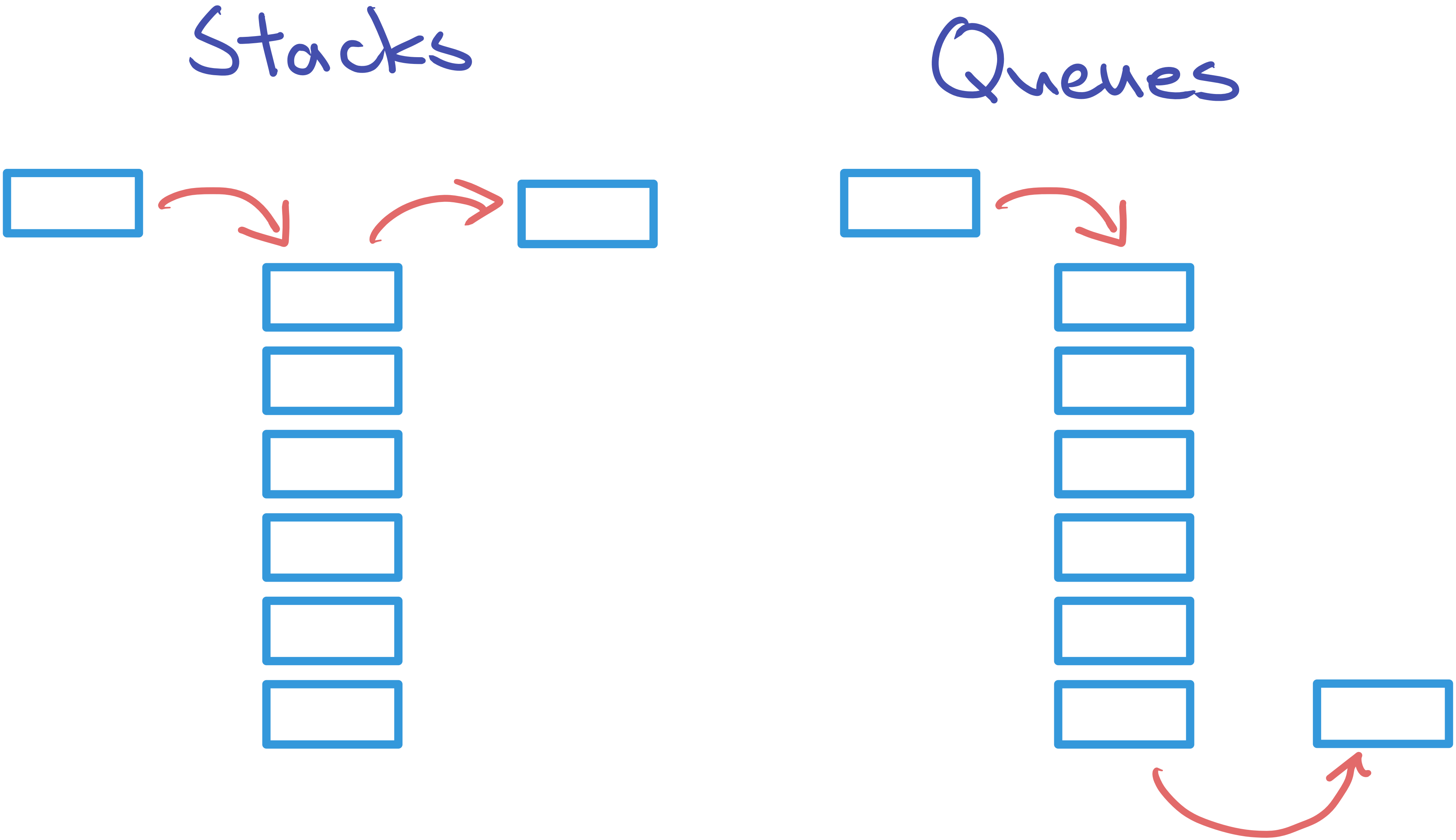
# Editing objects by copying

small change => copy whole object

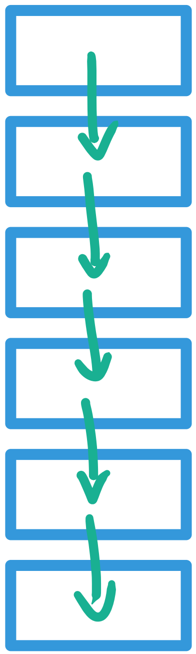


what about "deep" objects?



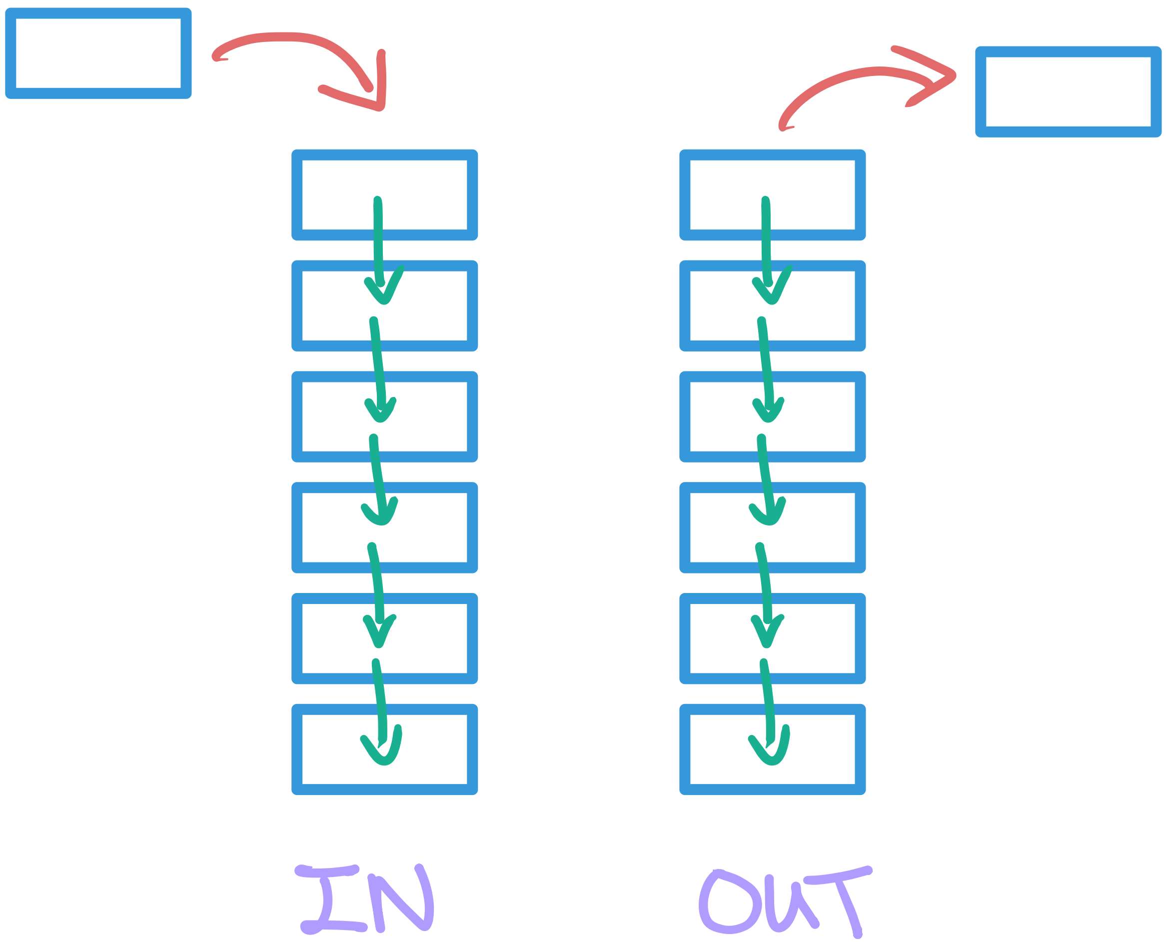

Stacks
Queues

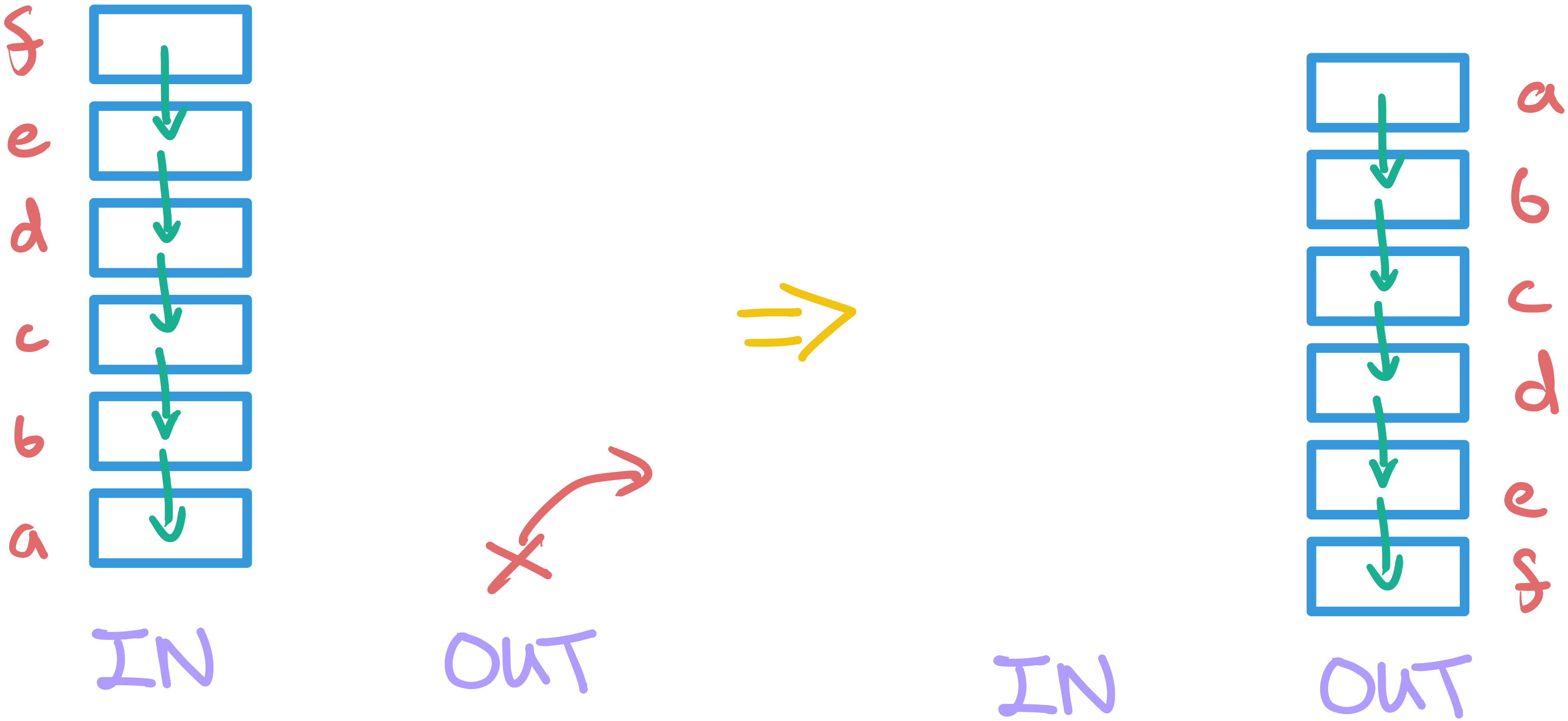
# Immutable stacks



(just a linked list)

# Immutable queues





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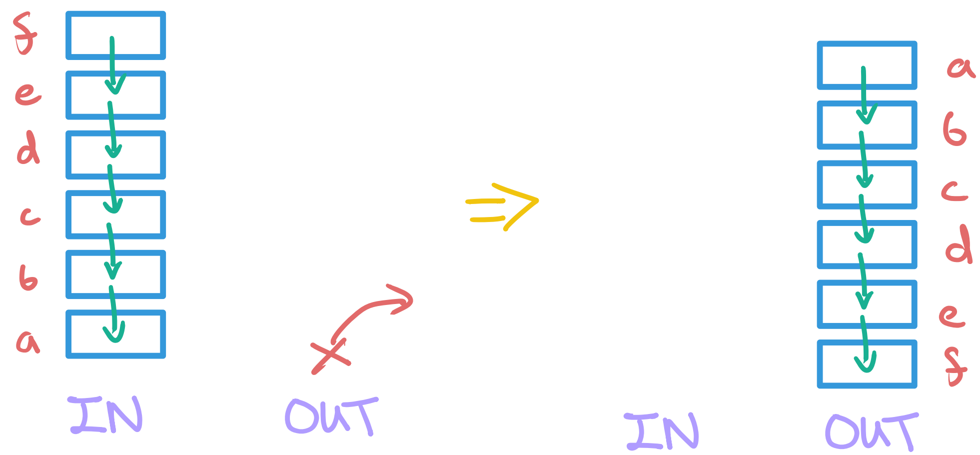


Theorem: insertion/deletion take $O(1)$ amortized